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FINITE STRIP—DIFFERENCE CALCULUS TECHNIQUE FOR PLATE VIBRATION PROBLEMS

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Abstract—The paper presents the application of the calculus of finite differences to obtain an explicit expression for the natural frequencies of the finite strip model of a simply supported orthotropic rectangular plate. This analytical solution not only involves far less computational work than the conventional finite strip method, but also enables parametric studies for accuracy and convergence of the finite strip approximation.

INTRODUCTION

The finite strip method[1], which is a Kantorovich type formulation of combining the finite element method and the Fourier series technique, has been used by many investigators for the free vibration analysis of plates[2–4]. This method requires less computer storage and time, compared to the finite element method, because of the reduced size of the stiffness and mass matrices involved in vibration analysis. However, as the number of strips increases the amount of computer work increases considerably.

For a rectangular plate of uniform thickness, simply supported along all the four edges, approximated as an assemblage of a number of finite strips of equal width, it is possible to obtain an analytical solution using the calculus of finite differences[5, 6]. The computational work involved is independent of the number of strips. Hence the finite strip-difference calculus technique can be used to test the accuracy of the finite strip approximation with increasing number of strips.§

STRIP STIFFNESS MATRIX

Consider a rectangular plate of uniform thickness and sides l and a simply supported along all the four edges. It is approximated as an assemblage of N_s strips (Fig. 1a). The lateral deflection of a typical strip of width d (Fig. 1b), defined by sides j and $(j+1)$ is assumed as

$$w(x,y) = \sum_{q=1,2\ldots}^{r} \left[\left(1-\frac{3x^2}{d^2}+\frac{2x^3}{d^3}\right) w_{j,q} + \left(x-\frac{2x^2}{d}+\frac{x^3}{d^2}\right)\theta_{j,q} + \left(\frac{3x^2}{d^2}-\frac{2x^3}{d^3}\right) w_{j+1,q} + \left(\frac{x^3}{d^2}-\frac{x^2}{d}\right)\theta_{j+1,q}\right] Y_q(y) \quad (1)$$

where the $Y_q(y)$ are functions which satisfy the boundary conditions of the strip at $y = 0$ and

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§A similar difference calculus approach has been used by Leckie [7] to test a Hrennikoff model approximation of plates.

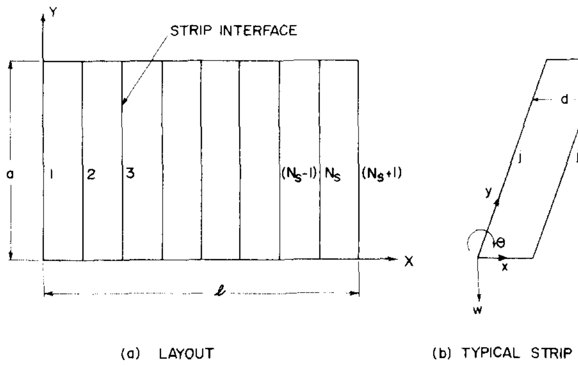

Fig. 1. Finite strip assemblage of the plate.

$y = a$, $w_{j,q}$ and $\theta_{j,q}$ are the lateral deflection and slope $(\partial w_{j,q}/\partial x)$ at the jth strip interface corresponding to $Y_q(y)$. Since the strips are simply supported, the $Y_q(y)$ may be taken as

$$Y_q(y) = \sqrt{\frac{2}{a}} \sin \frac{q\pi y}{a}. \tag{2}$$

Equation (1) can be written in the matrix form

$$w(x,y) = [\bar{N}]\{\bar{\Delta}\} = \sum_{q=1,2\ldots}^{r} [N]_q\{\Delta\}_q \tag{3}$$

where

$$\left.\begin{aligned} [N]_q &= \left[\left(1-\frac{3x^2}{d^2}+\frac{2x^3}{d^3}\right), \left(x-\frac{2x^2}{d}+\frac{x^3}{d^2}\right), \left(\frac{3x^2}{d^2}-\frac{2x^3}{d^3}\right), \left(\frac{x^3}{d^2}-\frac{x^2}{d}\right)\right] Y_q(y) \\ \{\Delta\}_q &= [w_{j,q} \quad \theta_{j,q} \quad w_{j+1,q} \quad \theta_{j+1,q}]^T \\ [\bar{N}] &= [[N]_1[N]_2 \ldots [N]_r] \\ \text{and} \\ \{\bar{\Delta}\} &= [[\Delta]_1{}^T[\Delta]_2{}^T \ldots [\Delta]_r{}^T]^T. \end{aligned}\right\} \tag{4}$$

The curvature matrix

$$\{\chi\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ +2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{5}$$

can be expressed in terms of the displacements as

$$\{\chi\} = [\bar{C}]\{\bar{\Delta}\} = \sum_{q=1,2\ldots}^{r} [C]_q\{\Delta\}_q. \tag{6}$$

Here

$$[\bar{C}] = [[C]_1[C]_2 \ldots [C]_r] \tag{7}$$

and

$$[C]_q = \begin{bmatrix} -\left(\frac{6}{d^2}+\frac{12x}{d^2}\right) Y_q & -\left(-\frac{4}{d}+\frac{6x}{d^2}\right) Y_q & -\left(\frac{6}{d^2}-\frac{12x}{d^3}\right) Y_q & -\left(\frac{6x}{d^2}-\frac{2}{d}\right) Y_q \\ -\left(1-\frac{3x^2}{d^2}+\frac{2x^3}{d^3}\right) Y_q'' & -\left(x-\frac{2x^2}{d}+\frac{x^3}{d^3}\right) Y_q'' & -\left(\frac{3x^2}{d^2}-\frac{2x^3}{d^3}\right) Y_q'' & -\left(\frac{x^3}{d^2}-\frac{x^2}{d}\right) Y_q'' \\ 2\left(-\frac{6x}{d^2}+\frac{6x^2}{d^3}\right) Y_q' & 2\left(1-\frac{4x}{d}+\frac{3x^2}{d^2}\right) Y_q' & 2\left(\frac{6x}{d^2}-\frac{6x^2}{d^3}\right) Y_q' & 2\left(\frac{3x^2}{d^2}-\frac{2x}{d}\right) Y_q' \end{bmatrix} \tag{8}$$

where

$$Y_q' = \frac{\mathrm{d}Y_q}{\mathrm{d}y}, \quad Y_q'' = \frac{\mathrm{d}^2 Y_q}{\mathrm{d}y^2}.$$

The bending and twisting moments for an orthotropic plate are given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D]\{\chi\} \tag{9}$$

where

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix}$$

in which D_x, D_y, D_1, D_{xy} are the flexural rigidities of the plate.

The strip stiffness matrix is given by

$$[\bar{k}] = \int_0^a \int_0^d [C]^T [D][C] \, \mathrm{d}x \, \mathrm{d}y \tag{10}$$

which yields

$$[\bar{k}] = \begin{bmatrix} [k]_{11} \; [k]_{12} \ldots [k]_{1r} \\ [k]_{21} \qquad \ldots \\ \cdot \\ \cdot \\ \cdot \\ [k]_{r1} \qquad \ldots [k]_{rr} \end{bmatrix} \tag{11}$$

where

$$[k]_{pq} = \int_0^a \int_0^d [\bar{C}]_p{}^T [D][\bar{C}]_q \, \mathrm{d}x \, \mathrm{d}y. \tag{12}$$

For strips simply supported at both ends with Y_q taken as in (2), it can be shown that

$$[k]_{pq} = 0 \text{ for } p \neq q. \tag{13}$$

$[k]_{qq}$ is given explicitly in Appendix 1.

STRIP MASS MATRIX

The consistent strip mass matrix has been used by earlier investigators for the plate vibration problems [2–4]. In this paper a concentrated line mass matrix, in which the distributed mass is assumed to be concentrated as line masses along the strip interfaces, is used because it is simpler to use in conjunction with the difference calculus technique.

Let μ be the uniform mass per unit area of the strip. For a typical strip the intensity of the line mass along its edges is $\mu(d/2)$ per unit length. So the inertial loading at the strip edges j and $(j+1)$ during free vibration is

$$\left.\begin{aligned} Q_j &= \mu \frac{d}{2} \omega^2 \sum_{q=1,2\ldots}^{r} w_{j,q} Y_{q(y)} \\ &\text{and} \\ Q_{j+1} &= \mu \frac{d}{2} \omega^2 \sum_{q=1,2\ldots}^{r} w_{j+1,q} Y_{q(y)} \end{aligned}\right\}. \tag{14}$$

Equation (14) can be written in terms of $[\bar{N}]$ and $\{\bar{\Delta}\}$ as

$$\left.\begin{aligned} Q_j &= \left(\mu \frac{d}{2} \omega^2 [\bar{N}]\{\bar{\Delta}\}\right)_{x=0} \\ &\text{and} \\ Q_{j+1} &= \left(\mu \frac{d}{2} \omega^2 [\bar{N}]\{\bar{\Delta}\}\right)_{x=d} \end{aligned}\right\}. \tag{15}$$

Using the principle of virtual work, as in the finite element method [8], the inertial force vector is obtained by

$$\{F\} = \int_0^a \int_0^d [\bar{N}]^T Q \, dx \, dy \tag{16}$$

where Q is the inertial force distribution. Expressing Q in terms of Q_j and Q_{j+1} and substituting in (16) gives

$$\{F\} = \mu \frac{d}{2} \omega^2 \int_0^a \int_0^d [\bar{N}]^T [\bar{N}]\{\bar{\Delta}\}(\Delta(x,\underline{o}) + \Delta(x,\underline{d}))dx \, dy \tag{17}$$

where $\Delta(x,\underline{o})$ is the Dirac delta function. We can also express the inertial forces in terms of the frequency, displacement and mass as

$$\{F\} = \omega^2 [\bar{m}]\{\bar{\Delta}\} \tag{18}$$

where $[\bar{m}]$ is the strip mass matrix. Comparing (17) and (18)

$$[\bar{m}] = \mu\frac{d}{2}\int_0^a\int_0^d [\bar{N}]^T[\bar{N}](\Delta(x,\underline{o}) + \Delta(x,\underline{d}))\,dx\,dy$$

$$= \mu\frac{d}{2}\int_0^a\int_0^d \begin{bmatrix} [N]_1^T[N]_1 & [N]_1^T[N]_2 & \dots & [N]_1^T[N]_r \\ [N]_2^T[N]_1 & \cdot & \dots & \cdot \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ [N]_r^T[N]_1 & \cdot & \dots & [N]_r^T[N]_r \end{bmatrix}$$

$$= \begin{bmatrix} [m]_{11} & [m]_{12} & \dots & [m]_{1r} \\ [m]_{21} & \cdot & \dots & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ [m]_{r1} & \cdot & \dots & [m]_{rr} \end{bmatrix} \tag{19}$$

where

$$[m]_{pq} = \mu\frac{d}{2}\int_0^a\int_0^d [N]_p^T[N_q](\Delta(x,\underline{o}) + \Delta(x,\underline{d}))\,dx\,dy$$

$$= \mu\frac{d}{2}\int_0^a ([1\,0\,0\,0]^T[1\,0\,0\,0] + [0\,0\,1\,0]^T[0\,0\,1\,0]Y_pY_q\,dy$$

$$= \mu\frac{d}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\int_0^a Y_pY_q\,dy. \tag{20}$$

Because of the orthonormal property of the Y_q taken in (2)

$$[m]_{pq} = 0 \text{ for } p \neq q \tag{21}$$

and

$$[m]_{qq} = \mu\frac{d}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

EQUILIBRIUM EQUATIONS FOR THE PLATE

The strip stiffness and mass matrices and the strip displacement vector are assembled to form the plate stiffness and mass matrices, $[\bar{K}]_{pq}$ and $[\bar{M}]_{pq}$, and the plate displacement vector $\{\bar{\delta}\}_q$. The boundary conditions in the x-direction are incorporated in the above matrices by deleting the rows and columns corresponding to the zero displacements. The resulting stiffness and mass matrices and displacement vector are $[K]_{pq}$, $[M]_{pq}$ and $\{\delta\}_q$ respectively.

Because of (13) and (21), we have

and

$$\left.\begin{aligned} [K]_{pq} &= 0 \text{ for } p \neq q \\ [M]_{pq} &= 0 \text{ for } p \neq q \end{aligned}\right\}. \tag{23}$$

So the equilibrium equation can be written mode by mode (i.e. for each integer value of q separately). The equilibrium equation for the qth mode is

$$[K]_{qq}\{\delta\}_q - \omega^2[M]_{qq}\{\delta\}_q = 0 \text{ for } q = 1,2,\ldots \tag{24}$$

The matrices $[K]_{qq}$ and $[M]_{qq}$ are of the order

$$N_0 = [2(N_s + 1) - \text{boundary conditions incorporated in the } x\text{-direction}].$$

THE FINITE STRIP METHOD

For the nontrivial solution of (24)

$$\mathrm{Det}([K]_{qq} - \omega^2[M]_{qq}) = 0. \tag{26}$$

This is a typical eigenvalue problem. The square of the frequencies of vibration, ω^2, are obtained as the eigenvalues of the matrix

$$[K]_{qq}^{-1}[M]_{qq}.$$

As the number of strips N_s increases the order of the above matrix, N_0, increases twice as fast and hence the computation involved also increases considerably.

FINITE STRIP-DIFFERENCE CALCULUS TECHNIQUE

Because of the repetitive pattern of the strip assemblage, it is possible to use the finite difference calculus [5, 6] in conjunction with the finite strip formulation for the frequency analysis of rectangular plates.

The equation of motion of the qth mode, before the incorporation of the boundary conditions in the x-direction (these will be incorporated later) is

$$([\bar{K}]_{qq} - \omega^2[\bar{M}]_{qq})\{\bar{\delta}\}_q = 0. \tag{27†}$$

This can be expanded as

$$\begin{bmatrix} s_{11}^{(1)} & s_{12}^{(1)} & s_{13}^{(1)} & s_{14}^{(1)} & 0 & 0 & \cdot & \cdot & \cdot \\ s_{21}^{(1)} & s_{22}^{(1)} & s_{23}^{(1)} & s_{24}^{(1)} & 0 & 0 & \cdot & \cdot & \cdot \\ s_{31}^{(1)} & s_{32}^{(1)} & s_{33}^{(1)} + s_{11}^{(2)} & s_{34}^{(1)} + s_{12}^{(2)} & s_{13}^{(2)} & s_{14}^{(2)} & \cdot & \cdot & \cdot \\ s_{41}^{(1)} & s_{42}^{(1)} & s_{43}^{(1)} + s_{21}^{(2)} & s_{44}^{(1)} + s_{22}^{(2)} & s_{23}^{(2)} & s_{24}^{(2)} & \cdot & \cdot & \cdot \\ 0 & 0 & s_{31}^{(2)} & s_{32}^{(2)} & s_{33}^{(2)} + s_{11}^{(3)} & s_{34}^{(2)} + s_{12}^{(3)} & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

†For the sake of simplicity, the suffix q will be dropped henceforth.

$$\begin{bmatrix} 0 & 0 & s_{41}^{(2)} & s_{42}^{(2)} & s_{43}^{(2)}+s_{21}^{(3)} & s_{44}^{(2)}+s_{22}^{(3)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} \theta_3 \\ \cdot \\ \cdot \end{Bmatrix} \begin{Bmatrix} 0 \\ \cdot \\ \cdot \end{Bmatrix} \tag{28}$$

where $s_{ij}^{(I)} = k_{ij}^{(I)} - \omega^2 m_{ij}^{(I)}$; $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$ and $I = 1, 2 \ldots N_s$ in which the superfix (I) denotes the number of the strip and $k_{ij}^{(I)}$, $m_{ij}^{(I)}$ are the elements of the strip stiffness and mass matrices $[k]_{qq}$ and $[m]_{qq}$ respectively.

Since all the strips have identical elastic and geometric properties, the mass and stiffness matrices are the same for all the strips, i.e.

$$\left.\begin{aligned} m_{ij}^{(I)} &= m_{ij} \\ k_{ij}^{(I)} &= k_{ij} \\ s_{ij}^{(I)} &= s_{ij} \ \text{ for } I = 1,2,\ldots N_S \end{aligned}\right\}. \tag{29}$$

Substituting (29) in (28) we find, on expanding the matrix equation into a system of simultaneous algebraic equations, that any pair of equations except the first and the last are similar. A typical pair of equations is

$$(k_{31}-\omega^2 m_{31})w_{j-1} + (k_{32}-\omega^2 m_{32})\theta_{j-1} + (k_{33}-\omega^2 m_{33} + k_{11} - \omega^2 m_{11})w_j$$
$$+ (k_{34}-\omega^2 m_{34} + k_{12} - \omega^2 m_{12})\theta_j + (k_{13}-\omega^2 m_{13})w_{j+1} + (k_{14}-\omega^2 m_{14})\theta_{j+1} = 0, \tag{30a}$$

$$(k_{41}-\omega^2 m_{41})w_{j-1} + (k_{42}-\omega^2 m_{42})\theta_{j-1} + (k_{21}-\omega^2 m_{21} + k_{43} - \omega^2 m_{43})w_j$$
$$+ (k_{22}-\omega^2 m_{22} + k_{44} - \omega^2 m_{44})\theta_j + (k_{23}-\omega^2 m_{23})w_{j+1} + (k_{24}-\omega^2 m_{24})\theta_{j+1} = 0. \tag{30b}$$

Because of the physical meaning of the stiffness coefficients, we have

$$\left.\begin{aligned} &k_{11} = k_{33}, \\ &k_{12} = k_{21} = -k_{34} = -k_{43}, \\ &k_{13} = k_{31}, \\ &k_{14} = k_{41} = -k_{23} = -k_{32}, \\ &k_{22} = k_{44} \\ \text{and}\quad &k_{24} = k_{42}. \end{aligned}\right\} \tag{31}†$$

The coefficients given in Appendix 1 verify equations (31).

Substituting (22) and (31) in (30a) and (30b) we get,

$$k_{13}(w_{j-1} + w_{j+1}) + k_{14}(\theta_{j+1} - \theta_{j-1}) + 2k_{11}w_j = \mu d\omega^2 w_j, \tag{32a}$$

$$k_{14}(w_{j-1} - w_{j+1}) + k_{24}(\theta_{j-1} + \theta_{j+1}) + 2k_{22}\theta_j = 0. \tag{32b}$$

†For example, k_{11} means the force required at the jth side of the strip to cause unit deflection (w) at the jth side. This is same as k_{33}, the force required at the $(j+1)$th side of the strip to cause unit deflection at the $(j+1)$th side.

Introducing the Boolean difference operator E, defined by

$$\left.\begin{aligned} Ew_j &= w_{j+1} \\ E^{-1}w_j &= w_{j-1} \end{aligned}\right\}, \quad (33)$$

and

equation (32) can be written as

$$[k_{13}(E^{-1}+E)+2k_{11}-\mu d\omega^2]w_j+[k_{14}(E-E^{-1})]\theta_j=0, \quad (34a)$$

$$[k_{14}(E^{-1}-E)]w_j+[k_{24}(E+E^{-1})+2k_{22}]\theta_j=0. \quad (34b)$$

Substituting (34b) in (34a), we get

$$\left\{[k_{13}(E+E^{-1})+2k_{11}-\mu d\omega^2]+\frac{[k_{14}^2(E-E^{-1})^2]}{[k_{24}(E+E^{-1})+2k_{22}]}\right\}w_j=0. \quad (35)$$

w_j may be assumed as

$$w_j=\sum_{n=1,2\ldots}A_n\phi_n(j) \quad (36)$$

where $\phi_n(j)$ satisfies the displacement boundary conditions at $x=0$ and $x=l$ (i.e. $j=0$ and $j=N_s$). Since the plate is simply supported at $x=0$ and $x=l$, a sinusoidal form of $\phi_n(j)$ may be assumed; i.e.

$$\phi_n(j)=\sin\frac{n\pi j}{N_s}. \quad (37)$$

Substituting (36) and (37) in (35) and using the relations

$$E\left\{\exp\left[\frac{in\pi j}{N_s}\right]\right\}=\exp\left[\frac{in\pi}{N_s}(j+1)\right]$$

and (38)

$$E^{-1}\left\{\exp\left[\frac{in\pi j}{N_s}\right]\right\}=\exp\left[\frac{in\pi}{N_s}(j-1)\right]$$

in which $i=\sqrt{-1}$, we get

$$\text{Imag.}\sum_{n=1,2\ldots}A_n\left\{\left\{k_{13}\left[\exp\left(\frac{in\pi}{N_s}\right)+\exp\left(\frac{-in\pi}{N_s}\right)\right]+2k_{11}-\mu d\omega^2\right\}\right.$$

$$\left.+\frac{\left\{k_{14}^2\left[\exp\left(\frac{in\pi}{N_s}\right)-\exp\left(\frac{-in\pi}{N_s}\right)\right]^2\right\}}{\left\{k_{24}\left[\exp\left(\frac{in\pi}{N_s}\right)+\exp\left(\frac{-in\pi}{N_s}\right)\right]+2k_{22}\right\}}\right\}\exp\left(\frac{in\pi j}{N_s}\right)=0 \quad (39)$$

where Imag. denotes the imaginary part. Simplifying (39), we get

$$\sum_{n=1,2\ldots} A_n\left\{\left[2k_{13}\cos\frac{n\pi}{N_s}+2k_{11}-\mu d\omega^2\right]-\frac{[2k_{14}^2\sin^2 n\pi/Ns]}{[k_{24}\cos n\pi/N_s+k_{22}]}\right\}\sin\frac{n\pi j}{N_s}=0. \tag{40}$$

So,

$$\left[2k_{13}\cos\frac{n\pi}{N_s}+2k_{11}-\mu d\omega^2\right]-\frac{[2k_{14}^2\sin^2 n\pi/N_s]}{[k_{24}\cos n\pi/N_s+k_{22}]}=0 \text{ for } n=1,2\ldots\infty. \tag{41}$$

From (41) the frequency ω_{qn} corresponding to the qth sine mode along the strip (Y-direction) and the nth sine mode across the strip (X-direction) is given by

$$\mu d\omega_{qn}^2=\frac{-[2k_{14}^2\sin^2 n\pi/N_s]}{[k_{24}\cos n\pi/N_s+k_{22}]}+2k_{13}\cos n\pi/N_s+2k_{11} \tag{42}$$

where the k_{ij} correspond the qth sine mode along the strip.

By changing the value of q and n the various frequencies can be computed. It is interesting to note that the computational work involved in calculating the above expression is independent of the number of strips, N_s, whereas it increases with N_s in the conventional finite strip method. Hence the present method is ideal to study the accuracy and convergence of finite strip approximation.

If the plate has boundary conditions other than simply supports at $x=0$ and $x=l$, then $\phi_n(j)$ can not be assumed as a sine function. When other appropriate functions are chosen, the resulting equations will not decouple in n as in equation (41). But a set of coupled linear algebraic equations will result and they have to be solved numerically. Hence a simple analytical expression for the frequencies, similar to (42), can not be obtained.

NUMERICAL RESULTS AND DISCUSSION

The natural frequencies of a rectangular simply supported plate of $l/a=2$, and $D_x=1$, $D_y=1$, $D_{xy}=0{\cdot}35$ and $D_1=0{\cdot}3$ are calculated by the conventional finite strip method [equation (26)] and the present method [equation (42)]. For different values of N_s, the approximate execution time required by an IBM 370-175 digital computer to calculate the (N_s-1) frequencies are given in Table 1. (In all cases q is taken equal to 1). It shows that as the number of strips, N_s, increases the execution time increases considerably for the conventional method, whereas the increase is very little for the present method.

Table 1. Execution time. (in 1/10,000 of a second)

N_s	Conventional method	Present method
5	798	432
10	1863	665
20	10549	865
30	35310	865

Squares of frequencies obtained from both the methods, along with the exact values†, are plotted in Fig. 2. The present method gives a lower bound in all the cases and converges towards

†For a simply supported plate, the exact solution is obtained by assuming

$$w(x,y)=\sum_{n=1,2\ldots}^{\infty}\sum_{q=1,2\ldots}^{\infty}B_{nq}\sin\frac{n\pi x}{l}\sin\frac{q\pi y}{a}.$$

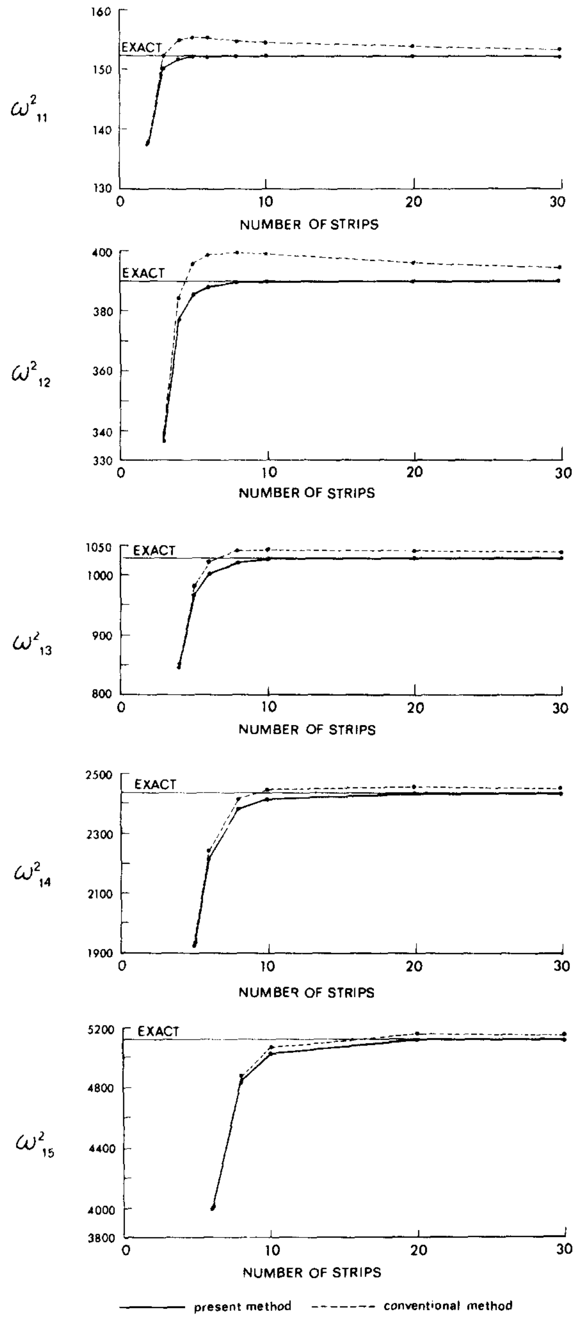

Fig. 2. Comparison of results.

the exact value as N_s increases. The conventional method gives lower values in the beginning increases above the exact value and then decreases towards the exact value. Moreover, in all cases the conventional method gives a higher value than the present method.

The conventional method is a computer-oriented eigenvalue analysis of equations (27), while the present method takes advantage of the repetitive pattern of the equations and uses the Calculus of Finite Differences to obtain analytical expressions for the eigenvalues. The differences in the results, as seen in Fig. 2 may be mainly due to the fact that the eigenvalues obtained by the conventional method are approximate due to errors in the numerical analysis of eigenvalues of a matrix, whereas the eigenvalues calculated from equation (42) are exact for the approximate strip-assembled plate model.

CONCLUSIONS

The finite strip-difference calculus technique proves to be a powerful tool to study the accuracy and convergence of the finite strip approximation. Whereas the conventional matrix technique gives approximate results, the present method gives an exact solution for the approximate structures, namely the strip-assemblage. Computational work is also considerably reduced in the present method.

The difference calculus technique for a plate vibration problem described in this paper, can be easily extended to static and stability problems of plates and shells.

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APPENDIX 1

$$[k]_{qq} = \left[\begin{array}{c|c|c|c}
\frac{12}{d^3}D_x+\frac{13}{35}dc^4D_y + \frac{12}{5d}c^2D_1+\frac{24}{5d}c^2D_{xy} & & & \\
\hline
\frac{6}{d^2}D_x+\frac{11d^2}{210}c^4D_y + \frac{6}{5}c^2D_1+\frac{2}{5}c^2D_{xy} & \frac{4}{d}D_x+\frac{2}{210}d^3c^4D_y + \frac{4}{15}dc^2D_1+\frac{8}{15}dc^2D_{xy} & \text{symmetrical} & \\
\hline
-\frac{12}{d^3}D_x+\frac{9}{70}dc^4D_y - \frac{12}{5d}c^2D_1-\frac{24}{5d}c^2D_{xy} & -\frac{6}{d^2}D_x+\frac{13}{420}d^2c^4D_y - \frac{1}{5}c^2D_1-\frac{2}{5}c^2D_{xy} & \frac{12}{d^3}D_x+\frac{13}{35}dc^4D_y + \frac{12}{5d}c^2D_1+\frac{24}{5d}c^2D_{xy} & \\
\hline
\frac{6}{d^2}D_x-\frac{13d^2}{420}c^4D_y + \frac{1}{5}c^2D_1+\frac{2}{5}c^2D_{xy} & \frac{2}{d}D_x-\frac{3}{420}d^3c^4D_y - \frac{1}{15}dc^2D_1-\frac{2}{15}dc^2D_{xy} & -\frac{6}{d^2}D_x-\frac{11}{210}d^2c^4D_y - \frac{6}{5}c^2D_1-\frac{2}{5}c^2D_{xy} & \frac{4}{d}D_x+\frac{1}{105}d^3c^4Dy + \frac{4}{15}dc^2D_1+\frac{8}{15}dc^2D_{xy}
\end{array}\right]$$

where $c = \left(\frac{q\pi}{a}\right)$.